

# Modified Chaplygin Gas and Solvable F-essence Cosmologies

Mubasher Jamil<sup>1</sup> • Yerlan Myrzakulov<sup>2</sup> •  
Olga Razina<sup>2</sup> • Ratbay Myrzakulov<sup>2,3</sup>

## Abstract

The Modified Chaplygin Gas (MCG) model belongs to the class of a unified models of dark energy and dark matter. In this paper, we have modeled MCG in the framework of f-essence cosmology. By constructing an equation connecting the MCG and the f-essence, we solve it to obtain explicitly the pressure and energy density of MCG. As special cases, we obtain both positive and negative pressure solutions for suitable choices of free parameters. We also calculate the state parameter which describes the phantom crossing.

**Keywords** Modified Chaplygin gas, K-essence, F-essence, G-essence.

## 1 Introduction

Several complementary cosmological observations guide us that our Universe is experiencing an accelerated expansion in the current era (Perlmutter et al 1999; Riess et al 1998). From the observations of Wilkinson Microwave Anisotropy Probe (WMAP) satellite which gathered data about the cosmic microwave background radiation, such a cosmic acceleration is produced by a so-called dark energy (DE) (Sherwin et al 2011). Such

a new element of the Universe, capable of accelerating, must be, in accordance with the Friedmann equation, have a pressure less than minus one third of the energy density. The notion of a cosmic medium possessing negative pressure is not new in cosmology. The very first proposal for dark energy is the cosmological constant denoted by  $\Lambda$ . Originally proposed by Einstein to construct a theoretical model of the universe such that the spherical configuration of matter in the universe is balanced with the negative pressure of cosmological constant. Thereby creating a static universe, in the conformity of the notions held by many scientists of that time. The origin of cosmic acceleration in recent time needs  $\Lambda$  but it faces serious problems of fine tuning (the value of  $\Lambda$  is several orders of magnitude larger than estimated from the empirical results) and cosmic coincidence problem (the energy density of matter and dark energy component are approximately same in our presence). Then theorists looked for other candidates of dark energy. There has been a wide variety of theoretical models of dark energy constructed in the literature including quintessence, phantom energy, Chaplygin gas, tachyon and dilaton dark energy etc, see (Copeland et al 2006) for further details. The quintessence and phantom energy models are based on spatially homogeneous and time dependent scalar fields, in conformity of the cosmological principle. The Lagrangians for the quintessence (phantom energy) has positive (negative) kinetic energy. Where quintessence faces still fine tuning problem of the parameters of its potential, while the phantom energy gives very esoteric possibilities of Big Rip and black hole evaporations (Jamil et al 2008).

Quintessence as a model of dark energy relies on the suitable choice of the potential function or the potential energy of scalar fields. It is also possible that the cosmic acceleration could appear due to modification of the kinetic energy of the

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Mubasher Jamil

Yerlan Myrzakulov

Olga Razina

Ratbay Myrzakulov

<sup>1</sup>Center for Advanced Mathematics and Physics, National University of Sciences and Technology, Islamabad, Pakistan. Email: mjamil@camp.nust.edu.pk

<sup>2</sup>Eurasian International Center for Theoretical Physics, Eurasian National University, Astana 010008, Kazakhstan

<sup>3</sup>Department of Physics, California State University, Fresno, CA 93740 USA, Email: rmyrzakulov@csufresno.edu, rmyrzakulov@gmail.com

scalar fields. Such modifications are termed non-canonical. The kinetically driven cosmic acceleration was originally proposed as a model for inflation, namely k-inflation (Armendariz-Picon et al 1999), and then as a model for dark energy, namely k-essence (Armendariz-Picon et al 2001, 2000; Chiba et al 2000; Chiba 2002; Scherrer 2004; Yang & Gao 2011; De Putter & Linder 2007; Capozziello et al 2010; Karami et al 2011; Khodam-Mohammadi & Tajiri 2010; Adabi et al 2011; Farooq et al 2010). This model is free from fine-tuning and anthropic arguments. K-essence has been proposed as a possible means of explaining the coincidence problem of the Universe beginning to accelerate only at the present epoch (Malquarti et al 2003). Instead, k-essence is based on the idea of a dynamical attractor solution which causes it to act as a cosmological constant only at the onset of matter-domination. Consequently, k-essence overtakes the matter density and induces cosmic acceleration at about the present epoch. In some models of k-essence, the cosmic acceleration continues forever while in others, it continues for a finite duration (Armendariz-Picon et al 2000).

In the last years, the k-essence model has received much attention. It is still worth investigating in a systematic way the possible cosmological behavior of the k-essence. Quite recently, a model named g-essence is proposed (Yerzhanov et al 2010a) which is a more generalized version of k-essence. In fact, the g-essence contains, as particular cases, two important models: k-essence and f-essence. Note that f-essence is the fermionic counterpart of k-essence.

To our knowledge, in the literature there are relatively few works on dark energy models with fermionic fields. However, in the recent years several approaches were made to explain the accelerated expansion by taking fermionic fields as the gravitational sources of energy (see e.g. refs. (Ribas et al 2005; Samojeden et al 2010, 2009; Myrzakulov 2010b; Tsyba et al 2011b; Yerzhanov et al 2010b; Ribas & Kremer 2010; Cai & Wang 2008; Wang et al 2010; Ribas et al 2008; Rakhi et al 2009, 2010; Chimento et al 2008; Anischenko et al 2009; Saha et al 2004; Saha & Shikin 1997; Saha 2001, 2004, 2006; Vakili & Sepangi 2008; Wei 2011; Dereli et al 2010; Balantekin & Dereli 2007; Armendariz-Picon & Greene 2003)). In particular, it was shown that the fermionic field plays very important role in: (i) isotropization of initially anisotropic space-time; (ii) formation of singularity free cosmological solutions; (iii) explaining late-time acceleration.

A very appealing proposal to describe the dark sector are the so-called unified models. The prototype of such model is the Chaplygin gas. In the unified models,

dark energy and dark matter are described by a single fluid, which behaves as ordinary matter in the past, and as a cosmological constant term in the future. In this sense, it interpolates the different periods of evolution of the Universe, including the present state of accelerated expansion. The Chaplygin gas model leads to very good results when confronted with the observational data of supernova type Ia. Concerning the matter power spectrum data, the statistic analysis leads to results competitive with the  $\Lambda$ CDM model, but the unified (called quartessence) scenario must be imposed from the beginning. It means that the only pressureless component is the usual baryonic one, otherwise there is a conflict between the constraints obtained from the matter power spectrum and the supernova tests. Note that many variations of the Chaplygin gas model have been proposed in the literature. One of them is the so-called Modified Chaplygin Gas model. It is important that the MCG model belongs to the class of a unified models of dark energy and dark matter. In this context, it is important to study the relation between MCG and the other unified models of dark energy and dark matter. For example, in (Myrzakulov 2011) relationship between MCG and k-essence was established. In this paper, we have modeled MCG in the framework of f-essence cosmology. By constructing an equation connecting the MCG and the f-essence, we solve it to obtain explicitly the pressure and energy density of MCG. As special cases, we obtain both positive and negative pressure solutions for suitable choices of free parameters.

This paper is organized as follows. In section II, we introduce the F-essence formalism. In section III, we briefly discuss Modified Chaplygin gas and its connection with the f-essence. In section IV, we construct a governing differential equation of our model and solve it for several special cases and a general case. Conclusion is presented in the last section. In the Appendix we provide the derivation of the equations of motion of g-essence, k-essence and f-essence.

## 2 F-essence

Let us briefly present some basics of f-essence. Its action has the form (Myrzakulov 2010b; Tsyba et al 2011b)

$$S = \int d^4x \sqrt{-g} [R + 2K(Y, \psi, \bar{\psi})], \quad (1)$$

where  $\psi$  and  $\bar{\psi} = \psi^\dagger \gamma^0$  denote the fermion field and its adjoint, respectively, the dagger represents complex conjugation and  $R$  is the Ricci scalar. The fermionic fields are treated here as classically commuting fields (see e.g. refs. (Ribas et al 2005;

Cai & Wang 2008; Wang et al 2010; Rakhi et al 2010; Chimento et al 2008; Saha et al 2004; Vakili & Sepangi 2008; Wei 2011; Dereli et al 2010; Balantekin & Dereli 2007; Armendariz-Picon & Greene 2003)). What we mean by a classical fermionic field is a set of four complex-valued space-time functions that transform according to the spinor representation of the Lorentz group. The existence of such fields is crucial in our work since despite the fact that fermions are described by quantized fermionic fields which do not have a classical limit, we assume such classical fields exist and use them as matter fields coupled to gravity. A possible justification for the existence of classical fermionic fields is given in the appendix of reference (Armendariz-Picon & Greene 2003). So for a more extensive and physically detailed discussion of the properties of such classical fermionic fields we refer to this fundamental work (Armendariz-Picon & Greene 2003). Furthermore,  $K$  is the Lagrangian density of the fermionic field, the canonical kinetic term has the form

$$Y = \frac{1}{2}i[\bar{\psi}\Gamma^\mu D_\mu\psi - (D_\mu\bar{\psi})\Gamma^\mu\psi]. \quad (2)$$

Moreover,  $\Gamma^\mu = e_a^\mu\gamma^a$  are the generalized Dirac-Pauli matrices satisfying the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (3)$$

where the braces denote the anti-commutation relation.  $e_a^\mu$  denotes the tetrad or "vierbein" while the covariant derivatives are given by

$$D_\mu\psi = \partial_\mu\psi - \Omega_\mu\psi, \quad D_\mu\bar{\psi} = \partial_\mu\bar{\psi} + \bar{\psi}\Omega_\mu. \quad (4)$$

Above, the fermionic connection  $\Omega_\mu$  is defined by

$$\Omega_\mu = -\frac{1}{4}g_{\rho\sigma}[\Gamma_{\mu\delta}^\rho - e_b^\rho\partial_\mu e_\delta^b]\Gamma^\sigma\Gamma^\delta, \quad (5)$$

with  $\Gamma_{\mu\delta}^\rho$  denoting the Christoffel symbols.

We work with the Friedmann-Robertson-Walker (FRW) spacetime given by

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2), \quad (6)$$

For this metric, the vierbein is chosen to be

$$(e_a^\mu) = \text{diag}(1, 1/a, 1/a, 1/a), \quad (e_\mu^a) = \text{diag}(1, a, a, a). \quad (7)$$

The Dirac matrices of curved spacetime  $\Gamma^\mu$  are

$$\Gamma^0 = \gamma^0, \Gamma^j = a^{-1}\gamma^j, \Gamma^5 = -i\sqrt{-g}\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \gamma^5, \quad (8)$$

$$\Gamma_0 = \gamma^0, \quad \Gamma_j = a\gamma^j, \quad (j = 1, 2, 3). \quad (9)$$

Hence we get

$$\Omega_0 = 0, \quad \Omega_j = 0.5\dot{a}\gamma^j\gamma^0 \quad (10)$$

and

$$Y = 0.5i(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi).$$

Finally, we note that the gamma matrices we write in the Dirac basis that is as

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad (11)$$

where  $I = \text{diag}(1, 1)$  and the  $\sigma^k$  are Pauli matrices having the following form

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (12)$$

We are ready to write the equations of f-essence (In detail, the derivation of these equation we will present in the Appendix). Here we write the final form of these equations. We have

$$3H^2 - \rho = 0, \quad (13)$$

$$2\dot{H} + 3H^2 + p = 0, \quad (14)$$

$$K_Y\dot{\psi} + 0.5(3HK_Y + \dot{K}_Y)\psi - i\gamma^0 K_{\bar{\psi}} = 0, \quad (15)$$

$$K_Y\dot{\bar{\psi}} + 0.5(3HK_Y + \dot{K}_Y)\bar{\psi} + iK_{\psi}\gamma^0 = 0, \quad (16)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (17)$$

where

$$\rho = YK_Y - K, \quad p = K, \quad (18)$$

are the energy density and pressure of the fermionic field. If  $K = Y - V$ , then from the system (13)-(18) we get the corresponding equations of the Einstein-Dirac model. At last, we note that f-essence is the particular case of g-essence for which the energy density and pressure are given by

$$\rho = 2XK_X + YK_Y - K, \quad p = K, \quad (19)$$

where  $X = 0.5\dot{\phi}^2$  is the canonical kinetic term for the scalar field  $\phi$  (see the Appendix).

### 3 Modified Chaplygin gas and f-essence

In the cosmological context, the Chaplygin gas was first suggested as an alternative to quintessence and

was demonstrated to have an increasing  $\Lambda$  (cosmological constant) behavior for the evolution of the universe (Kamenshchik et al 2001; Bento et al 2002). The EoS of the MCG dark energy model was proposed by Benaoum (Benaoum 2002) as an exotic fluid which could explain the cosmic accelerated expansion. Later on, it was shown that the EoS of MCG is valid from the radiation era to  $\Lambda$ CDM model (Debnath et al 2004). The MCG parameters  $\alpha$  and  $B$  have been constrained by the cosmic microwave background CMB data (Liu & Li 2005). The stable scaling solutions (attractor) of the Freidmann equation have been obtained in (Jamil & Rashid 2008, 2009). The MCG is given by (Benaoum 2002)

$$p = A\rho - \frac{B}{\rho^\alpha}, \quad (20)$$

where  $A$  and  $B$  are positive constants and  $0 \leq \alpha \leq 1$ . In the light of 3-year WMAP and the SDSS data, the MCG best fits the data by choosing  $A = -0.085$  and  $\alpha = 1.724$  (Lu et al 2008). The dynamical attractor for the MCG exists at  $\omega = -1$  (where  $p = \omega\rho$ ), hence MCG can do the phantom crossing from  $\omega > -1$  to  $\omega < -1$ , independent to the choice of initial conditions (Jing et al 2008). A generalization of MCG was proposed in (Debnath 2007) by taking  $B \equiv B(a) = B_0 a^n$ , where  $n$  and  $B_0$  are constants. The MCG is the generalization of generalized Chaplygin gas  $p = -B/\rho^\alpha$  (Barreiro & Sen 2004; Carturan & Finelli 2003) with the addition of a barotropic term. Using Eqs. (17) and (20), we can show that the MCG energy density and pressure are given by (Benaoum 2002)

$$\rho = \left[ B(1+A)^{-1} + C a^{-3(1+\alpha)(1+A)} \right]^{\frac{1}{1+\alpha}}, \quad (21)$$

$$p = [AB(1+A)^{-1} + AC a^{-3(1+\alpha)(1+A)}] - \frac{B}{\rho^\alpha} \left[ B(1+A)^{-1} + C a^{-3(1+\alpha)(1+A)} \right]^{-\frac{\alpha}{1+\alpha}}, \quad (22)$$

where  $C$  is a constant of integration. This constant of integration  $C$  can be found from the condition that the fluid has vanishing pressure  $p = 0$  when  $a = a_0$ :

$$C = B[A(1+A)]^{-1} a_0^{3(1+\alpha)(1+A)}.$$

For the modified Chaplygin gas case, the EoS parameter is

$$\omega = \frac{AC a^{-3(1+\alpha)(1+A)} - B(1+A)^{-1}}{B(1+A)^{-1} + C a^{-3(1+\alpha)(1+A)}}. \quad (23)$$

From (18) we get

$$\rho = \frac{dK}{d \ln Y} - K, \quad (24)$$

$$p = K. \quad (25)$$

Hence follows that

$$\rho + p = \frac{dp}{d \ln Y} = \frac{dp}{da} \frac{da}{d \ln Y}$$

so that we have

$$\frac{d \ln Y}{da} = \frac{p_a}{p + \rho}, \quad (26)$$

where  $p_a = \frac{dp}{da}$ . The last equation has the following solution

$$Y = Y_0 e^{\int \frac{p_a}{p+\rho} da}, \quad (27)$$

where  $Y_0$  is an integration constant. We can rewrite this expression as

$$Y = Y_0 e^{\int \frac{p_\zeta}{p+\rho} d\zeta}, \quad (28)$$

where  $\zeta = C a^{-3(1+\alpha)(1+A)}$ . Then the corresponding expressions for the energy density and pressure take the form

$$\rho = [D + \zeta]^{\frac{1}{1+\alpha}}, \quad (29)$$

$$p = [A(D + \zeta) - B] [D + \zeta]^{-\frac{\alpha}{1+\alpha}} \quad (30)$$

with  $D = B(1+A)^{-1}$ . Then we have

$$p + \rho = (1+A)\zeta(D + \zeta)^{-\frac{\alpha}{1+\alpha}}, \quad (31)$$

$$p_\zeta = A(D + \zeta)^{-\frac{\alpha}{1+\alpha}} - \frac{\alpha}{1+\alpha} (A\zeta - D)(D + \zeta)^{-\frac{\alpha}{1+\alpha}-1}. \quad (32)$$

Hence we get

$$\int \frac{p_\zeta}{p + \rho} d\zeta = \ln \left[ \zeta^{\frac{(1+\alpha)A+\alpha}{(1+\alpha)(1+A)}} (D + \zeta)^{-\frac{\alpha}{1+\alpha}} \right]. \quad (33)$$

Finally we have

$$Y = Y_0 \zeta^{\frac{(1+\alpha)A+\alpha}{(1+\alpha)(1+A)}} (D + \zeta)^{-\frac{\alpha}{1+\alpha}}. \quad (34)$$

## 4 Solvable f-essence cosmologies

In this section, we consider some solvable f-essence models related with the MCG given by (20).

### 4.1 The case: $B = 0$

In this case the EoS takes the form

$$p = A\rho, \quad (35)$$

where  $\rho$  evolves like

$$\rho = \rho_0 a^{-3(1+A)}, \quad \rho_0 = \text{const} \quad (36)$$

so that

$$p = A\rho_0 a^{-3(1+A)}. \quad (37)$$

On the other hand, the equations (24)-(25) for (35) give

$$\rho = FY^{\frac{1+A}{A}}, \quad F = \text{const.} \quad (38)$$

In this case, the pressure is

$$p \equiv K = AFY^{\frac{1+A}{A}}. \quad (39)$$

Comparison of (36) and (38) yields

$$Y = Y_0 a^{-3A}, \quad Y_0 = \rho_0^{\frac{A}{1+A}} F^{-\frac{A}{1+A}}. \quad (40)$$

We get the scale factor as

$$a(t) = \left(\frac{Y}{Y_0}\right)^{\frac{-1}{3A}}. \quad (41)$$

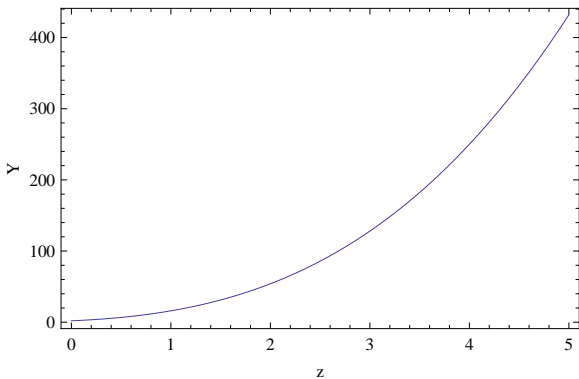
The behavior of Eq. (40) against redshift  $a^{-1} - 1 = z$  is plotted in figure-1 where we see that this evolution is of power-law form.

#### 4.2 The case: $A = 0$

Ignoring the barotropic term in MCG, we have

$$p = -\frac{B}{\rho^\alpha}. \quad (42)$$

It is called the generalized Chaplygin gas (GCG) (Kamenshchik et al 2001). Recently it is proposed using perturbative analysis and power spectrum observational data that the MCG model is not a successful candidate for the cosmic medium unless  $A = 0$ , i.e. the usual GCG model is favored (Fabris et al 2011).



**Fig. 1** Evolution of kinetic energy  $Y$  as a function of redshift  $z$  from Eq. (40). Other parameters are fixed at  $Y_0 = 2$  and  $A = -1$ .

As well-known, the corresponding energy density and pressure are given by

$$\rho = \left[B + Ca^{-3(1+\alpha)}\right]^{\frac{1}{1+\alpha}}, \quad (43)$$

$$p = -B \left[B + Ca^{-3(1+\alpha)}\right]^{-\frac{\alpha}{1+\alpha}}, \quad (44)$$

where  $C$  is a constant of integration. From (34), (43) and (44) we get the expressions for the energy density and pressure:

$$\rho = \left[\frac{B}{1 - (WY)^{\frac{1+\alpha}{\alpha}}}\right]^{\frac{1}{1+\alpha}}, \quad (45)$$

$$p = -B^{\frac{1}{1+\alpha}} \left[1 - (WY)^{\frac{1+\alpha}{\alpha}}\right]^{\frac{\alpha}{1+\alpha}}, \quad (46)$$

where  $W = \text{const.}$  The solution for  $Y$  is determined from (43) and (45):

$$Y = W^{-1} C^{\frac{\alpha}{1+\alpha}} \left[C + Ba^{3(1+\alpha)}\right]^{-\frac{\alpha}{1+\alpha}}. \quad (47)$$

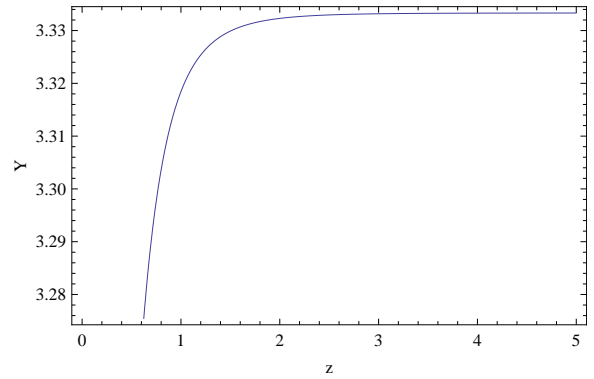
The behavior of Eq. (47) is shown in figure-2, where we see that the kinetic energy of the f-essence increases and then stays constant at higher redshifts. Note that this conclusion depends crucially on the chosen values of free parameters.

#### 4.3 The general case

In this section, we consider the general case when  $A \neq 0$ ,  $B \neq 0$ . So in this case we must solve the following system

$$\rho = YK_Y - K, \quad (48)$$

$$K = A\rho - \frac{B}{\rho^\alpha} \quad (49)$$



**Fig. 2** Evolution of  $Y$  against  $z$  from Eq. (47). Other parameters are fixed at  $C = 0.5$ ,  $B = 0.4$ ,  $W = 0.3$  and  $\alpha = 1.2$ .

or

$$\rho = Y p_Y - p, \quad (50)$$

$$p = A\rho - \frac{B}{\rho^\alpha}. \quad (51)$$

Solving equation (48) for  $K$ , we arrive at

$$p \equiv K = EY + Y \int \frac{\rho}{Y^2} dY, \quad (52)$$

where  $E$  is an integration constant. Note that if  $\rho = V(\bar{\psi}, \psi)$  then from (52), it follows that  $K = EY - V(\bar{\psi}, \psi)$  i.e. the purely Dirac case. Eqs. (51) and (52) give

$$EY + Y \int \frac{\rho}{Y^2} dY = A\rho - \frac{B}{\rho^\alpha}, \quad (53)$$

which has the following solution:

$$(1+A)\rho^{1+\alpha} - (WY)^{\frac{n(1+\alpha)}{\alpha}} \rho^{n(1+\alpha)} - B = 0, \quad (54)$$

where  $W$  is a constant and

$$n = \frac{\alpha(1+A)}{A + \alpha(1+A)}. \quad (55)$$

From (55) it follows that  $A$  and  $\alpha$  are related by

$$A = -\frac{(n-1)\alpha}{(n-1)\alpha + n} \quad \text{or} \quad \alpha = -\frac{nA}{(n-1)(1+A)}. \quad (56)$$

The search of the analytical solutions of Eq. (54) is a tough job. So let us find some particular solutions for some values of  $n$ .

#### 4.3.1 Example 1: $n = 0$

It follows from (55) that this case ( $n = 0$ ) realized as  $\alpha = 0$  or  $A = -1$ . 1) Let us first consider the case  $\alpha = 0$ . Then  $n = 0$  and the equations (48)-(49) take the form

$$\rho = Y K_Y - K, \quad (57)$$

$$K = A\rho - B \quad (58)$$

and the equation (54) becomes

$$(1+A)\rho^{1+\alpha} - (WY)^{\frac{1+A}{A}} - B = 0. \quad (59)$$

Hence we write

$$\rho = (1+A)^{-1} [B + (WY)^{\frac{1+A}{A}}]. \quad (60)$$

and for the pressure

$$p = (1+A)^{-1} [-B + A(WY)^{\frac{1+A}{A}}]. \quad (61)$$

The corresponding equation of state (EoS) parameter is given by

$$\omega = A - \frac{B(1+A)}{B + (WY)^{\frac{1+A}{A}}}. \quad (62)$$

2) Now we consider the case when  $A = -1$ . Then  $n = 0$  and equations (48)-(49) take the form

$$\rho = Y K_Y - K, \quad (63)$$

$$K = -\rho - B\rho^{-\alpha}. \quad (64)$$

Hence we get

$$\alpha B \ln \rho - (1+\alpha)^{-1} \rho^{1+\alpha} = \ln(C_1 Y)^{-B}. \quad (65)$$

Consider some particular solutions of this equation. If  $\alpha = 0$ , then we have

$$\rho = \ln(C_1 Y)^B, \quad (66)$$

and for the pressure

$$p = \ln(C_1 Y)^{-B} - B. \quad (67)$$

The corresponding EoS parameter is given by

$$\omega = -1 - [\ln(C_1 Y)]^{-1}. \quad (68)$$

Second example is  $\alpha = -1$ . Then for the energy density and pressure we obtain

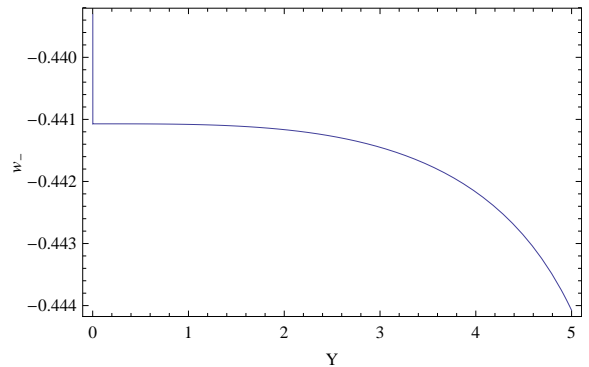
$$\rho = C_2 Y^{\frac{B}{1+B}} \quad (C_2 = \text{const}), \quad (69)$$

and

$$p = -(1+B)C_2 Y^{\frac{B}{1+B}}. \quad (70)$$

The corresponding EoS parameter is given by

$$\omega = -1 - B. \quad (71)$$



**Fig. 3** For  $n = 2$ , the EoS parameter  $w_-$  is plotted against the kinetic energy  $Y$ , while other parameters are fixed at  $B = 0.1$ ,  $W = 0.2$ ,  $\alpha = 1.5$



#### 4.3.2 Example 2: $n = 1$

If  $n = 1$  then from (56), it follows that  $A = 0$ . This case was considered in subsection 4.2, so we omit it here.

#### 4.3.3 Example 3: $n = 2$

Now we consider the case when  $n = 2$ . In this case  $A$  and  $\alpha$  related by

$$A = -\frac{\alpha}{\alpha + 2} \quad \text{or} \quad \alpha = -\frac{2A}{1 + A}. \quad (72)$$

The equation for  $\rho$  (54) takes the form

$$(1 + A)\rho^{1+\alpha} - (WY)^{\frac{2(1+\alpha)}{\alpha}}\rho^{2(1+\alpha)} - B = 0. \quad (73)$$

It has the solution

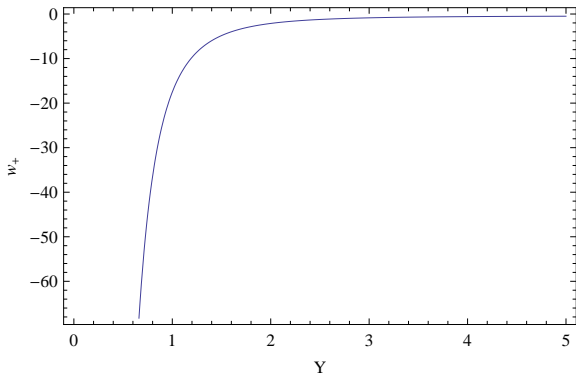
$$\rho = (WY)^{-\frac{2}{\alpha}} \left\{ \frac{1}{1 + \alpha} \times \left[ 1 \mp \sqrt{1 - B(1 + \alpha)^2 (WY)^{\frac{2(1+\alpha)}{\alpha}}} \right] \right\}^{\frac{1}{1+\alpha}}. \quad (74)$$

The pressure is given by

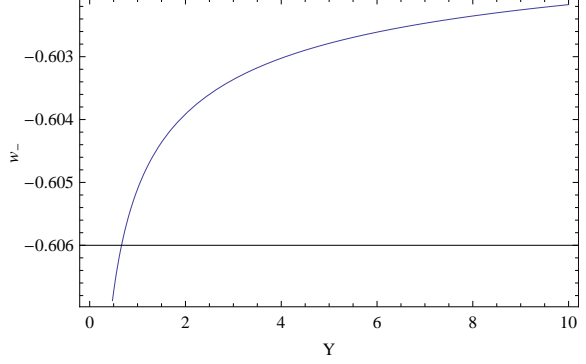
$$p = -\frac{\alpha(WY)^{-\frac{2}{\alpha}}}{\alpha + 2} \left\{ \frac{1}{1 + \alpha} \times \left[ 1 \mp \sqrt{1 - B(1 + \alpha)^2 (WY)^{\frac{2(1+\alpha)}{\alpha}}} \right] \right\}^{\frac{1}{1+\alpha}} - B(WY)^2 \left\{ \frac{1}{1 + \alpha} \times \left[ 1 \mp \sqrt{1 - B(1 + \alpha)^2 (WY)^{\frac{2(1+\alpha)}{\alpha}}} \right] \right\}^{-\frac{\alpha}{1+\alpha}}. \quad (75)$$

The corresponding EoS parameter is

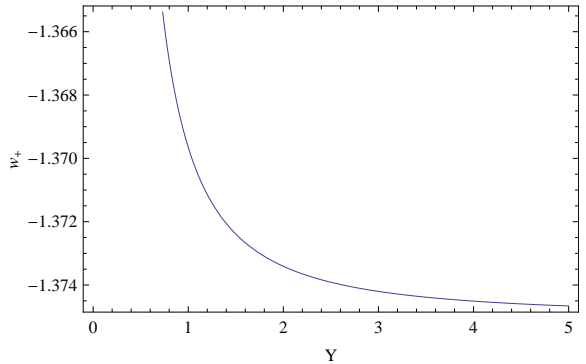
$$\omega_{\mp} = -\frac{\alpha}{\alpha + 2} - B(WY)^{-\frac{2(1+\alpha)}{\alpha}} \left\{ \frac{1}{1 + \alpha} \times \left[ 1 \mp \sqrt{1 - B(1 + \alpha)^2 (WY)^{\frac{2(1+\alpha)}{\alpha}}} \right] \right\}. \quad (76)$$



**Fig. 4** For  $n = 2$ , the EoS parameter  $w_+$  is plotted against the kinetic energy  $Y$ , while other parameters are fixed at  $B = 0.1$ ,  $W = 0.2$ ,  $\alpha = 1.5$



**Fig. 5** For  $n = 0.5$ , the EoS parameter  $w_-$  is plotted against the kinetic energy  $Y$ , while other parameters are fixed at  $B = -0.1$ ,  $W = 0.2$ ,  $\alpha = -1.5$



**Fig. 6** For  $n = -1$ , the EoS parameter  $w_+$  is plotted against the kinetic energy  $Y$ , while other parameters are fixed at  $B = -1$ ,  $W = 2$ ,  $\alpha = 1.5$

In figures (3) and (4), we have plotted the EoS parameter and it is shown that subnegative values of  $\omega$  are permissible in our model. This corresponds to f-essence MCG behaving as phantom energy which causes super-accelerated expansion (Debnath et al 2004).

#### 4.3.4 Example 4: $n = 0.5$

If  $n = 0.5$  then  $A$  and  $\alpha$  satisfy the relation

$$A = -\frac{\alpha}{\alpha - 1} \quad \text{or} \quad \alpha = \frac{A}{1 + A}. \quad (77)$$

Eq. (54) becomes

$$(1 + A)\rho^{1+\alpha} - (WY)^{\frac{(1+\alpha)}{2\alpha}}\rho^{0.5(1+\alpha)} - B = 0. \quad (78)$$

This equation has the following solution

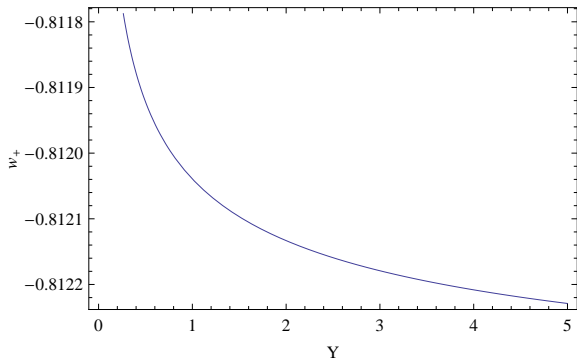
$$\rho = (WY)^{\frac{1}{\alpha}} \left\{ \frac{1 - \alpha}{2} \left[ 1 \pm \sqrt{1 + \frac{4B}{1 - \alpha}(WY)^{-\frac{1+\alpha}{\alpha}}} \right] \right\}^{\frac{2}{1+\alpha}}. \quad (79)$$

The pressure is given by

$$\begin{aligned} p = & -\frac{\alpha(WY)^{\frac{1}{\alpha}}}{\alpha - 1} \left\{ \frac{1 - \alpha}{2} \right. \\ & \times \left[ 1 \pm \sqrt{1 + \frac{4B}{1 - \alpha}(WY)^{-\frac{1+\alpha}{\alpha}}} \right] \left. \right\}^{\frac{2}{1+\alpha}} \\ & - B(WY)^{-1} \left\{ \frac{1 - \alpha}{2} \right. \\ & \times \left[ 1 \pm \sqrt{1 + \frac{4B}{1 - \alpha}(WY)^{-\frac{1+\alpha}{\alpha}}} \right] \left. \right\}^{-\frac{2\alpha}{1+\alpha}}. \end{aligned} \quad (80)$$

The corresponding EoS parameter reads

$$\begin{aligned} \omega_{\mp} = & -\frac{\alpha}{\alpha - 1} - \frac{(WY)^{\frac{1+\alpha}{\alpha}}}{1 - \alpha} \\ & \times \left[ 1 \mp \sqrt{1 + \frac{4B}{1 - \alpha}(WY)^{-\frac{1+\alpha}{\alpha}}} \right]^2. \end{aligned} \quad (81)$$



**Fig. 7** For  $n = -1$ , the EoS parameter  $w_+$  is plotted against the kinetic energy  $Y$ , while other parameters are fixed at  $B = -3$ ,  $W = 0.8$ ,  $\alpha = -1.5$

In figure-5, we have plotted the EoS parameter against the kinetic term. Here we chose  $\alpha < 0$  which corresponds to the polytropic term added in the barotropic equation of state. Such a f-essence polytropic EoS can also cause the super-acceleration.

#### 4.3.5 Example 5: $n = -1$

Our next example is  $n = -1$ . Then  $A$  and  $\alpha$  satisfy the relation

$$A = -\frac{2\alpha}{2\alpha - 1} \quad \text{or} \quad \alpha = \frac{A}{2(1 + A)}. \quad (82)$$

Eq. (54) becomes

$$(1 + A)\rho^{1+\alpha} - (WY)^{-\frac{1+\alpha}{\alpha}}\rho^{-(1+\alpha)} - B = 0. \quad (83)$$

It has the solution

$$\rho = \left[ \frac{B \pm \sqrt{B^2 + 4(1 + A)(WY)^{-\frac{1+\alpha}{\alpha}}}}{2(1 + A)} \right]^{\frac{1}{1+\alpha}}. \quad (84)$$

The pressure is given by

$$\begin{aligned} p = & A \left[ \frac{B \pm \sqrt{B^2 + 4(1 + A)(WY)^{-\frac{1+\alpha}{\alpha}}}}{2(1 + A)} \right]^{\frac{1}{1+\alpha}} \\ & - B \left[ \frac{B \pm \sqrt{B^2 + 4(1 + A)(WY)^{-\frac{1+\alpha}{\alpha}}}}{2(1 + A)} \right]^{-\frac{\alpha}{1+\alpha}}. \end{aligned} \quad (85)$$

The corresponding EoS parameter is given by

$$\begin{aligned} \omega_{\mp} = & -\frac{2\alpha}{2\alpha - 1} + \frac{B}{2}(WY)^{\frac{1+\alpha}{\alpha}} \\ & \times \left[ B \mp \sqrt{B^2 - \frac{1}{2\alpha - 1}(WY)^{-\frac{1+\alpha}{\alpha}}} \right]. \end{aligned} \quad (86)$$

In figures 6 and 7, we plotted the above state parameter against kinetic energy. We choose positive and negative values of  $\alpha$  for the sake of completeness. It is apparent that the state parameter achieves subnegative values showing the viability of our dark energy model.

## 5 Conclusion

In summary, we have modeled modified Chaplygin gas in f-essence cosmology. The use of MCG is useful as a tool of explaining dark energy and dark matter in a unified manner, while f-essence cosmology essentially suitable to describe cosmic acceleration only at present time. Thus the correspondence of MCG with f-essence is useful in learning how these two models are connected to each other. We studied this link by constructing a



differential equation (54) connecting the MCG and the f-essence. We solved it to obtain explicitly the pressure and energy density of MCG. We observed that f-essence MCG has one additional free parameter namely  $W$  along with  $A$ ,  $B$ ,  $\alpha$ . As special cases, we obtain both positive (37), (80) and negative (46), (68), (72) pressure solutions for suitable choices of free parameters. The negative pressure solution is essentially useful for cosmic expansion with acceleration. Prior to this, we studied the model with barotropic and generalized Chaplygin gas equation of states.

The present work is concerned only with the correspondence of MCG with the f-essence. However it would be more interesting to study the dynamical features of this model. For instance, it is interesting to check whether such a model is useful to model inflation at earlier cosmic epoch and cosmic acceleration at the present time. Next it will be important to compare this model with the observational data and constrain its free parameters, particularly  $W$ . Also observational constraints on the state parameter  $\omega$  will indicate if it corresponds to cosmological constant, quintessence or phantom energy. We finish our work in the hope that merits and demerits of f-essence will be clear in the future investigations.

We note that this paper is the logical continuation of (Myrzakulov 2011), where the relation between k-essence and MCG was studied. The action of k-essence reads as

$$S = \int d^4x \sqrt{-g} [R + 2K(X, \phi)], \quad (87)$$

where the kinetic term  $X$  for the scalar field  $\phi$  (for the FRW metric) reads as

$$X = 0.5\dot{\phi}^2. \quad (88)$$

The corresponding equations for the FRW metric look like

$$3H^2 - \rho = 0, \quad (89)$$

$$2\dot{H} + 3H^2 + p = 0, \quad (90)$$

$$K_X \ddot{\phi} + (\dot{K}_X + 3HK_X)\dot{\phi} - K_\phi = 0, \quad (91)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (92)$$

where the energy density and pressure of the scalar field is given by

$$\rho = 2XK_X - K, \quad p = K. \quad (93)$$

In this case, the common system of equations for the k-essence and MCG has the form

$$\rho = 2Xp_X - p, \quad (94)$$

$$p = A\rho - \frac{B}{\rho^\alpha}. \quad (95)$$

The compatibility condition for the equations (94)-(95) is given by (Myrzakulov 2011)

$$(1 + A)\rho^{1+\alpha} - (WX)^{\frac{n(1+\alpha)}{2\alpha}}\rho^{n(1+\alpha)} - B = 0, \quad (96)$$

where  $W = W(\phi)$  and

$$n = \frac{\alpha(1 + A)}{A + \alpha(1 + A)}. \quad (97)$$

In (Myrzakulov 2011), different type solvable k-essence cosmologies compatible with the MCG model are found for the different values of  $n$ .

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## 6 APPENDIX. The derivation of the equations of motion of g-essence, k-essence and f-essence

In this Appendix we would like to present the derivation of the equations of motion for the f-essence action (2.1) that is the system (13)-(17). But, as f-essence is the exact particular case of g-essence, we consider more general case and give the derivation of the equations of motion for g-essence. Let us consider the following action of g-essence

$$S = \int d^4x \sqrt{-g} [R + 2K(X, Y, \phi, \psi, \bar{\psi})], \quad (98)$$

where  $R$  is the scalar curvature,  $X$  is the kinetic term for the scalar field  $\phi$ ,  $Y$  is the kinetic term for the fermionic field  $\psi$  and  $K$  is some function (Lagrangian) of its arguments. In the case of the FRW metric

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2), \quad (99)$$

$R$ ,  $X$  and  $Y$  have the form

$$R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right), \quad (100)$$

$$X = 0.5\dot{\phi}^2, \quad (101)$$

$$Y = 0.5i(\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi), \quad (102)$$

respectively. Substituting (100)-(102) into (98) and integrating over the spatial dimensions, we are led to an effective Lagrangian in the mini-superspace  $\{a, \phi, \psi, \bar{\psi}\}$

$$L = -2(3a\dot{a}^2 - a^3K). \quad (103)$$

Variation of Lagrangian (103) with respect to  $a$  yields the equation of motion of the scale factor

$$2a\ddot{a} + \dot{a}^2 + a^2K = 0. \quad (104)$$

Now by varying the above Lagrangian (103) with respect to the scalar field  $\phi$  we obtain its equation of motion as

$$K_X\ddot{\phi} + \dot{K}_X\dot{\phi} + 3\frac{\dot{a}}{a}K_X\dot{\phi} - K_\phi = 0. \quad (105)$$

At least, the variation of Lagrangian (103) with respect to  $\bar{\psi}, \psi$  that is the corresponding Euler-Lagrangian equations for the fermionic fields give

$$K_Y\gamma^0\dot{\psi} + 1.5\frac{\dot{a}}{a}K_Y\gamma^0\psi + 0.5\dot{K}_Y\gamma^0\psi - iK_{\bar{\psi}} = 0, \quad (106)$$

$$K_Y\dot{\bar{\psi}}\gamma^0 + 1.5\frac{\dot{a}}{a}K_Y\bar{\psi}\gamma^0 + 0.5\dot{K}_Y\bar{\psi}\gamma^0 + iK_\psi = 0. \quad (107)$$

Also, we have the "zero-energy" condition given by

$$L_a\dot{a} + L_\phi\dot{\phi} + L_\psi\dot{\psi} + L_{\bar{\psi}}\dot{\bar{\psi}} - L = 0 \quad (108)$$

which yields the constraint equation

$$-3a^{-2}\dot{a}^2 + 2XK_X + YK_Y - K = 0. \quad (109)$$

Collecting all derived equations (104) - (107) and (109) and rewriting them using the Hubble parameter  $H = (\ln a)_t$ , we come to the following closed system of equations of g-essence (for the FRW metric case):

$$3H^2 - \rho = 0, \quad (110)$$

$$2\dot{H} + 3H^2 + p = 0, \quad (111)$$

$$K_X\ddot{\phi} + (\dot{K}_X + 3HK_X)\dot{\phi} - K_\phi = 0, \quad (112)$$

$$K_Y\dot{\psi} + 0.5(3HK_Y + \dot{K}_Y)\psi - i\gamma^0K_{\bar{\psi}} = 0, \quad (113)$$

$$K_Y\dot{\bar{\psi}} + 0.5(3HK_Y + \dot{K}_Y)\bar{\psi} + iK_\psi\gamma^0 = 0, \quad (114)$$

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (115)$$

Here

$$\rho = 2XK_X + YK_Y - K, \quad p = K \quad (116)$$

are the energy density and pressure of g-essence. It is clear that these expressions for the energy density and pressure represent the components of the energy-momentum tensor of g-essence:

$$T_{00} = 2XK_X + YK_Y - K, \quad T_{11} = T_{22} = T_{33} = -K.$$

(117)

Also we note that g-essence admits two particular cases (reductions): k-essence and f-essence. In fact, let  $\psi = 0$ . Then  $Y = 0$  and the system (110)-(115) becomes

$$3H^2 - \rho = 0, \quad (118)$$

$$2\dot{H} + 3H^2 + p = 0, \quad (119)$$

$$K_X\ddot{\phi} + (\dot{K}_X + 3HK_X)\dot{\phi} - K_\phi = 0, \quad (120)$$

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (121)$$

where

$$\rho = 2XK_X - K, \quad p = K. \quad (122)$$

It is the system of equations of k-essence. Now let us consider the case when  $\phi = 0$  that is the purely fermionic case. Then  $X = 0$ ,  $K = K(Y, \bar{\psi}, \psi)$  and the system (110)-(115) takes the form of the equations of f-essence that is (13)-(17).

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